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B.Sc. Part - I, Paper - I

Theory of Equations (Cardon's Solution of a Cubic)

The Cardon's solution, which is similar to the perfect-square method to quadratic equations, is a standard way to find a real root of a cubic equation like

$$ax^3 + bx^2 + cx + d = 0$$

we can then find the other two roots (real or complex) by polynomial division and the quadratic formula. The solution has two steps. we first "depress" the cubic equation and then solve the depressed equation.

To solve the depressed equation of the form $y^3 + Ay = B$

we first substitute

$$3st = A, \quad s^3 - t^3 = B.$$

Then we solve for

$$y = s - t.$$

Example. Find the real root of the cubic equation

$$2x^3 - 30x^2 + 162x - 350 = 0$$

Soln From the above example, we know that the given equation is depressed into $y^3 + 6y = 20$. Then we substitute

$$3st = 6 \quad \text{———} \quad (1)$$

$$s^3 - t^3 = 20 \quad \text{———} \quad (2)$$

from (1) we have $s = \frac{2}{t}$, substituting which into

(2) gives
$$\frac{8}{t^3} - t^3 = 20 \quad \text{—— (2a)}$$
multiplying (2a) throughout by t^3, we obtain the quadratic equation for t^3 as follows:
$$t^6 + 20t^3 - 8 = 0$$
Solving for t^3 and the t, we have
$$t^3 = \frac{-20 \pm 12\sqrt{3}}{2} \Rightarrow t = \sqrt[3]{-10 \pm 6\sqrt{3}}$$
It can be shown that, ~~whatever~~ whether we take the positive or negative root of t, we will get the same value for $y = s - t$. we shall only consider the positive $t = \sqrt[3]{-10+6\sqrt{3}}$

from (2) we have $s = \sqrt[3]{20-10+6\sqrt{3}} = \sqrt[3]{10+6\sqrt{3}}$, which implies
$$y = s - t = \sqrt[3]{10+6\sqrt{3}} - \sqrt[3]{-10+6\sqrt{3}} = 2.$$
Therefore, the real root of the original cubic equation $2x^3 - 30x^2 + 162x - 350 = 0$ is
$$x = y + 5 = 2 + 5 = 7.$$

When we check that $x = 7$ is the only real root of the cubic equation $2x^3 - 30x^2 + 162x - 350 = 0$ because $2x^3 - 30x^2 + 162x - 350$ can be factorized as
$$2x^3 - 30x^2 + 162x - 350$$
$$2(x-7)(x^2 - 8x + 25).$$
[The cubic poly $P : ax^3 + bx^2 + cx + d = 0$ has solutions
$$x_1 = S + T - \frac{b}{3a}, \quad x_2 = -\frac{S+T}{2} - \frac{b}{3a} + \frac{i\sqrt{3}}{2}(S-T)$$
$$x_3 = -\frac{S+T}{2} - \frac{b}{3a} - \frac{i\sqrt{3}}{2}(S-T), \text{ where } S = \sqrt[3]{R + \sqrt{Q^3+R^2}},$$
$$T = \sqrt[3]{R - \sqrt{Q^3+R^2}}, \quad Q = \frac{3ac - b^2}{9a^2}, \quad R = \frac{9abc - 27a^2d - 2b^3}{54a^3}.\ ]$$